Lecture 2 on Sept.09 2013

In this lecture, we studied the geometric representation of the complex numbers. if we are given $\alpha + i\beta$ a complex number, here α and β are real numbers representing the associated real and imaginary parts respectively, then we can construct a map from $\mathbb{C}$ to $\mathbb{R}^2$ by

$$\alpha + i\beta \longmapsto (\alpha, \beta).$$

In fact, we identify complex numbers with points in $\mathbb{R}^2$. Under this map, the addition for complex numbers corresponds to vector addition in $\mathbb{R}^2$. Moreover by polar coordinates in $\mathbb{R}^2$, we can rewrite $\alpha + i\beta$ as follows:

$$\alpha + i\beta = r(\cos\theta + i\sin\theta),$$

where r is the absolute value of $\alpha + i\beta$, θ is the angle between (α, β) and positive direction of x-axis in $\mathbb{R}^2$. Conventionally we call θ the argument of $\alpha + i\beta$. Suppose that we are given

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_2), \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

then one can easily show that

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)) \\ &= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)). \end{aligned}$$

In other word, the multiplication between two complex numbers corresponds to multiplying norms and adding arguments.

The polar representation of complex numbers gives us a easy way to calculate n-th roots of a complex number. Assume that $z = r(\cos\theta + i\sin\theta)$ satisfies

$$z^n = a = r_0(\cos\theta_0 + i\sin\theta_0).$$

Then we know that $r^n(\cos n\theta + i\sin n\theta) = r_0(\cos\theta_0 + i\sin\theta_0)$. Hence we deduce that

$$r^n = r_0, \qquad n\theta = \theta_0 + 2k\pi, \quad k \in \mathbb{Z}.$$

Solving the above equations gives us

$$r = r_0^{1/n}, \qquad \theta = \frac{\theta_0}{n} + \frac{2k\pi}{n}, \quad k \in \mathbb{Z}.$$

Varying k within integer numbers can give us n different solutions of $z^n = a$.

Analytically using complex numbers can also give us a easy way to represent lines and circles in $\mathbb{R}^2$ space. Given a point $a = (a_1, a_2)$ in $\mathbb{R}^2$, then the line passing this point and pointing to the direction (b_1, b_2) can be easily written as $a+bt$. Here $a = a_1+ia_2$, $b = b_1+ib_2$. t is a real parameter running from $-\infty$ to ∞. A circle with center (a_1, a_2) and radius r can be written as $|z - a| = r$. Here $a = a_1 + ia_2$. $|\cdot|$ stands for the absolute value for complex numbers.

Now we study the second geometric representation of complex numbers. The so-called Riemann sphere. Firstly, we embed the complex plane to the plane P, which is located at $x_3 = 0$ in $\mathbb{R}^3$. Therefore $z = a + ib$ can be represented by the point $(a, b, 0)$ on the plane. Denoting by $\mathbb{S}^2$ the unit sphere in $\mathbb{R}^3$, then we can find out its north pole, denoted by $N = (0, 0, 1)$. By $(a, b, 0)$ and $(0, 0, 1)$, we can construct a line in $\mathbb{R}^3$. This line must have one intersection with $\mathbb{S}^2$, say (x_1, x_2, x_3). We call (x_1, x_2, x_3) the stereographic projection of $a + ib$. Here we have constructed a one-one correspondence between $\mathbb{C}$ and $\mathbb{S}^2 \setminus \{N\}$. One can easily represent (x_1, x_2, x_3) by the given point $a + ib$. In fact we have

Proposition 1. *If (x_1, x_2, x_3) is the stereographic projection of $a + ib$, then*

$$x_1 = \frac{2a}{1 + a^2 + b^2}, \qquad x_2 = \frac{2b}{1 + a^2 + b^2}, \qquad x_3 = \frac{a^2 + b^2 - 1}{a^2 + b^2 + 1}. \tag{0.1}$$

Conversely, if we have a point (x_1, x_2, x_3) on $\mathbb{S}^2$, then it represents $a + ib$ on the complex plane with

$$a = \frac{x_1}{1 - x_3}, \qquad b = \frac{x_2}{1 - x_3}. \tag{0.2}$$

Proposition 2. *From (0.1), we see that if $|z| = \sqrt{a^2 + b^2} \longrightarrow \infty$, then*

$$|x_1|, |x_2| \leq \frac{2|z|}{1 + |z|^2} \longrightarrow 0, \qquad x_3 \longrightarrow 1.$$

The above proposition implies that no matter how a point diverges to ∞ on $\mathbb{C}$, the associated projection on $\mathbb{S}^2$ will always converge to the north pole. Therefore we can identify ∞ on $\mathbb{C}$ with N on $\mathbb{S}^2$. Therefore we obtain a one-one correspondence between $\mathbb{C} \cup \{\infty\}$ and $\mathbb{S}^2$. In the following, we study two important properties of the stereographic projection. The first one is

Theorem 1. *The stereographic projection maps all circles on $\mathbb{S}^2$ to circles or lines on the complex plane $\mathbb{C}$.*

The proof of this theorem can be found from the textbook. On the other hand, we also have

Theorem 2. *all lines and circles on $\mathbb{C}$ are mapped to circles on $\mathbb{S}^2$ by stereographic projection.*

Theorems 1 and 2 tell us that there is a one-one correspondence between circles on $\mathbb{S}^2$ and circles on $\mathbb{C}$. Here we regard lines on $\mathbb{C}$ as circles with radius ∞. In the following, we consider the proof of Theorem 2. It just involves a little bit vector calculus. Suppose we have a line $a + tb$. Since this line contains ∞ on $\mathbb{C}$, N must be lied in the image of $a + tb$ under the stereographic projection. Pick up t_1, t_2, t_3 three real numbers, then we can find $a + t_1 b$, $a + t_2 b$, $a + t_3 b$ three points on $\mathbb{C}$. They have three images on $\mathbb{S}^2$ under the stereographic projection. They are given by

$$P_j = \left(\frac{2(\mathrm{Re}a + t_j \mathrm{Re}b)}{1 + |a + t_j b|^2}, \frac{2(\mathrm{Im}a + t_j \mathrm{Im}b)}{1 + |a + t_j b|^2}, \frac{|a + t_j b|^2 - 1}{1 + |a + t_j b|^2} \right), \quad j = 1, 2, 3.$$

$P_j - N$ give us three vectors on $\mathbb{R}^3$. By calculating determinant, one can easily show that these three vectors are linearly dependent. In other words , they lie on the same plane. Since t_j are arbitrarily chosen, the image of $a + tb$ must be a circle. Circles on $\mathbb{C}$ are mapped to circles on $\mathbb{S}^2$. This fact is left as an exercise.

The second property is that the stereographic projection is conformal. Equivalently it preserves angles. The proof of this fact is not required in this course. But I still write it out in the following for your reference. The readers may refer to the graphic in another file.

Proof of the conformal property of stereographic projection. Suppose N is the north pole, P is point on $\mathbb{S}^2$. $\pi(P)$ is the projection of P under the stereographic projection. Let T be the tangent line of P which lies in the plane ONP. T intersect with O $\pi(P)$ at W. Let γ be a path on $\mathbb{S}^2$. P is contained in γ. The projection of γ is denoted by $\pi(\gamma)$. At point W, we draw a line l which is orthogonal to the plane ONP. γ determines a tangent line at P, say l_1. Clearly T, l_1 and l are all orthogonal to the line OP. Therefore l_1 must lie in the plane (P, l). Here (P, l) denotes the plane containing P and l. So we have two choices. Either l_1 parallels to l or intersects with l at some point Q. We consider the second case.

Now on the sphere $\mathbb{S}^2$, we have two paths across P. One is the big circle C, which is determined by intersecting $\mathbb{S}^2$ with ONP. Another path is the γ. Under the stereographic projection, the images of these two paths are the line OW and $\pi(\gamma)$, respectively. Therefore the problem is reduced to show that the angle between γ and C equals to the angle between $\pi(\gamma)$ and OW. the angle between γ and C is $\angle QPW$ since by our construction l_1 is tangent to γ while T is tangent to C. To consider the angle between $\pi(\gamma)$ and OW, we need find out the tangent line of $\pi(\gamma)$ at $\pi(P)$. We claim that $Q\pi(P)$ is the tangent line of $\pi(\gamma)$ at $\pi(P)$. Therefore the angle between $\pi(\gamma)$ and OW is $\angle Q\pi(P)W$. Now we just need to show that

$$\angle QPW = \angle Q\pi(P)W. \tag{0.3}$$

Noticing that $\angle QWP = \angle QW\pi(P) = \pi/2$, QW is the common edge of the two triangles QWP and $QW\pi(P)$. Moreover one can easily show that the length of PW equals to the length of $W\pi(P)$. Therefore the two triangles QWP and $QW\pi(P)$ are identical, which shows (0.3). The proof is finished. □